Calculations of the Mechanical Tolerances for the BetaCage

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1 Introduction

This note describes several calculations on how the mechanical tolerances can affect the gain of the central anode grids of both the bulk and trigger frames in the BetaCage. Hopefully, this will settle the discussion on whether the feed through design can be modified to have a 100 μm diameter hole to hold the 25 μm diameter wire instead of the original 50 μm design.

The Diethorn formula (reproduced from Ref. [1]) for calculating the gain, $G$, is given by

$$\ln G = \frac{\ln 2}{\Delta V} \frac{\lambda}{2\pi \epsilon_0} \ln \frac{\lambda}{2\pi \epsilon_0 r E_{\text{min}} \rho / \rho_0},$$

(1)

where $\epsilon_0 = 8.854 \times 10^{-15}$ F/mm is the vacuum permittivity, $\lambda$ is the linear charge density on a wire, $r$ is the radius of the wire, $\rho / \rho_0$ is the gas pressure relative to standard pressure, $\Delta V$ is the average potential difference needed to produce one or more electrons, and $E_{\text{min}}$ is the minimum field needed for ionization at standard pressure. Clearly, both $\Delta V$ and $E_{\text{min}}$ are dependent on the specific gas. For a mixture of argon and methane the values are given in Table 1. For the following calculations these numbers are averaged as they are not that different.

<table>
<thead>
<tr>
<th>Mixture</th>
<th>$\Delta V$ [V]</th>
<th>$E_{\text{min}}$ [kV/mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ar(90%) + CH₄(10%)</td>
<td>23.6±5.4</td>
<td>4.8±0.3</td>
</tr>
<tr>
<td>Ar(95%) + CH₄(5%)</td>
<td>21.8±4.4</td>
<td>4.4±0.4</td>
</tr>
</tbody>
</table>

Mechanical variations affect the gain by changing the charge density on the wires. To study the size of the gain variations, the Diethorn formula (Eqn. 1) is differentiated with respect to charge to yield

$$\frac{\partial G}{G} = \left[ \ln G + \frac{\lambda \ln 2}{2\pi \epsilon_0 \Delta V} \right] \frac{\partial \lambda}{\lambda}.$$  

(2)

This derivation is different than that found in Sauli [2], where they are missing the second term in the brackets. As this term is on the same order as $\ln G$ in some configurations, they underestimate the effect of charge variations.

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2 Capacitance Matrix

The solution to the electrostatics problem can be simplified by assuming that the wire grids are infinite in extent. The potential of a wire grid relative to a grounded conductor is given by

\[ V(x, z) = \frac{-\lambda}{4\pi\epsilon_0} \ln \left( \frac{\sin^2[(\pi/s)(x-x_0)] + \sinh^2[(\pi/s)(z-z_0)]}{\sin^2[(\pi/s)(x-x_0)] + \sinh^2[(\pi/s)(z+z_0)]} \right) \] (3)

where \( s \) is the pitch of the wire grid, \( x_0 \) is the \( x \) (lateral) position of a wire, and \( z_0 \) is the \( z \) (vertical) position of the wire grid. At distances greater than or comparable to the pitch of the plane the grid behaves as a uniform sheet with surface charge density \( \lambda/s \). This implies

\[ V(x, z) = \frac{\lambda z}{s\epsilon_0} \quad z < z_0 \] (4)

\[ V(x, z) = \frac{\lambda z_0}{s\epsilon_0} \quad z > z_0 \] (5)

for \( |z - z_0| \gg s/2\pi \). These expressions are easily verified by substituting the hyperbolic expressions by exponentials. The potential at the surface of a wire, and therefore the potential for the entire grid, can be calculated by demanding that \( (x-x_0)^2 + (z-z_0)^2 \equiv r^2 \) where \( r \) is the wire radius, and noting that both \( (x-x_0) \) and \( (z-z_0) \) are much less than \( s/\pi \). The trigonometric quantities in \( qnb \) can be expanded to first order, and the hyperbolic term in \( (z+z_0) \approx 2z_0 \) can be approximated by a positive exponential giving

\[ V(\text{wire}) \approx \frac{\lambda}{4\pi\epsilon_0} \ln \left( \frac{e^{4\pi z_0/s}4}{(\pi/s)^2 r^2} \right) = \frac{\lambda z_0}{s\epsilon_0} \left( 1 - \frac{s}{2\pi z_0} \ln \frac{2\pi r}{s} \right), \] (6)

the potential at the wire due to its charge. The logarithmic term is certainly negative for this approximation to hold, thus the potential is always the same sign as the charge.

For the bulk frame there are three wire grids some distance away from a grounded plane (i.e. the inner cathode of the trigger grid). Figure 1 shows a sketch of the geometry. For the current BetaCage geometry

\[
\begin{pmatrix}
V_c \\
V_a \\
V_c
\end{pmatrix}
= \frac{1}{\epsilon_0}
\begin{pmatrix}
(z_0 + \Delta z)(1 - \frac{s}{2\pi(z_0 + \Delta z)} \ln \frac{2\pi r_c}{s}) & z_0 \frac{s}{2\pi z_0} \ln \frac{2\pi r_a}{s} & z_0 - \Delta z \\
\frac{s}{2\pi z_0} \ln \frac{2\pi r_a}{s} & (z_0 - \Delta z)(1 - \frac{s}{2\pi(z_0 - \Delta z)} \ln \frac{2\pi r_c}{s}) & z_0 - \Delta z
\end{pmatrix}
\]

Figure 1: Edge-on view of the BetaCage geometry. The grounded plane is meant to represent the trigger grid. The top wire grid and bottom wire grids are held at potential \( V_c \) and are positioned 90° in the \( x-y \) plane relative to the central anode wire grid. The anode grid is held at potential \( V_a \).

\( z_0 = 205 \text{ mm} \), and \( \Delta z = s = 5 \text{ mm} \). With these numbers, and the expressions given by Eqns. 4, 5, and 6, a matrix can be formed relating the surface charges \( \lambda/s \) to the applied voltages by superposition giving

\[ \left( \begin{array}{c} V_{c1} \\ V_a \\ V_{c2} \end{array} \right) = \frac{1}{\epsilon_0} \left( \begin{array}{ccc} (z_0 + \Delta z)(1 - \frac{s}{2\pi(z_0 + \Delta z)} \ln \frac{2\pi r_c}{s}) & z_0 \frac{s}{2\pi z_0} \ln \frac{2\pi r_a}{s} & z_0 - \Delta z \\ z_0 \frac{s}{2\pi z_0} \ln \frac{2\pi r_a}{s} & (z_0 - \Delta z)(1 - \frac{s}{2\pi(z_0 - \Delta z)} \ln \frac{2\pi r_c}{s}) & z_0 - \Delta z \end{array} \right) \]
where the labels $c1$, $a$, and $c2$ represent the top cathode, middle anode, and bottom cathode grids respectively. The $3 \times 3$ matrix (along with $1/\epsilon_0$) is recognized as the inverse of the capacitance matrix, or $C^{-1}$. The fact that each wire grid acts like an infinite sheet of charge beyond a distance $s/2\pi$ has allowed for the consideration of wire grids with differing $x - y$ plane orientations. The capacitance matrix is calculated to be

$$C = \frac{\epsilon_0}{\text{mm}} \begin{pmatrix} 0.108 & -0.073 & -0.034 \\ -0.073 & 0.147 & -0.073 \\ -0.034 & -0.073 & 0.111 \end{pmatrix},$$

which is symmetric as required by reciprocity.

In order to calculate the gain, the linear charge densities are needed. For voltages of $V_{c1} = V_{c2} = 1000$ V and $V_a = 3000$ V the charge densities are (from inverting Eqn. 7) $-6.47 \times 10^{-12}$ C/mm, $13.03 \times 10^{-12}$ C/mm, and $-6.27 \times 10^{-12}$ C/mm for the top cathode, middle anode, and bottom cathode grids respectively. Putting this all together, the gain (Eqn. 1) on the anode grid is $G = 2.1330 \times 10^4 = 10^{4.329}$. The calculation of the capacitance matrix for the trigger grid is slightly simpler in that the matrix is $2 \times 2$. Granted, this is assuming that the capacitive coupling between the trigger frame and bulk frame is small (see appendix A for the full $5 \times 5$ capacitance matrix). The capacitance matrix for the trigger frame is

$$C = \frac{\epsilon_0}{\text{mm}} \begin{pmatrix} 0.114 & -0.066 \\ -0.066 & 0.161 \end{pmatrix},$$

where the first row and column is the inner grounded cathode grid and the second row and column is the trigger-frame anode grid. A gain of $G = 10^5$ is produced for an anode voltage of $V_a = 2000$ V and grounded cathodes.

### 3 Variation of the anode wires diameter

The variation of an anode wire’s diameter will mostly likely occur because of a manufacturing defect. In principle, this should affect all the wires in the grid. Remember the expression for the voltage is

$$V = C^{-1} \cdot \sigma,$$

where $\sigma \equiv \lambda/s$. Then for the case of constant potentials the variation of the charge is

$$\partial \lambda = -C \cdot \partial C^{-1} \cdot \lambda,$$

where the pitch has been multiplied out to change from charge per area to charge per length. As only one term in $C^{-1}$ is a function of $r_a$, the derivative is simple yielding

$$\frac{\partial \lambda_a}{\lambda_a} = C_{22} \frac{s}{2 \pi \epsilon_0 \, r_a} \frac{\partial r_a}{r_a} = 0.117 \frac{\partial r_a}{r_a}.$$

This implies that it would take a $5\%$ change in radius to equal a $10\%$ change in gain. For a single wire diameter variation, the dependence is nearly independent of geometry and found to be [3]

$$\frac{\Delta \lambda}{\lambda} \approx 0.2 \frac{\Delta r_a}{r_a},$$

where a $3\%$ change in the radius of a single wire changes the gain by $10\%$. 

3
4 Displacement of the bulk frame

Because of the block-frame construction of the BetaCage, overall variations of anode plane between cathodes or of the cathode-to-cathode distance are unlikely, and much smaller than the distance between the three grids. What is more likely, is a displacement of the bulk frame relative to the trigger frame. This can be calculated by differentiating the charge versus the frame center, $z_0$.

$$\frac{\partial \lambda_a}{\lambda_a} = -\frac{1}{\epsilon_0} \left[ \sum_{m=1}^{3} C_{2m} \right] \sum_{\lambda} \lambda \frac{\partial z}{\lambda_a},$$

(14)

where $\sum \lambda$ is the sum of the charge densities. As both sums in this equation are near zero, this effect is tiny and ignorable. For instance, a 1 cm change in frame position changes the gain by 0.2%.

5 Displacement of a single wire

The displacement of a single wire is somewhat more complicated than the previous discussions. This is because the simplifications that led to a $3 \times 3$ capacitance matrix no longer apply. For a full treatment of the problem see Ref. [3]. Here, the results are summarized.

The discussion considers the capacitive coupling between every wire of the wire grid while the grid is sandwiched between two grounded conductors. An iterative approach is applied to calculate the values of arbitrary coefficients of the infinite series. Doing so finds the fractional charge variation versus fractional displacement in $x$ and $z$ and is given by [3]

$$\frac{\Delta \lambda_0}{\lambda} \approx a_2^{(0)} \left( \frac{\Delta x}{s} \right)^2 + b_2^{(0)} \left( \frac{\Delta z}{s} \right)^2,$$

(15)

$$\frac{\Delta \lambda_j}{\lambda} \approx a_1^{(j)} \left( \frac{\Delta x}{s} \right) + b_2^{(j)} \left( \frac{\Delta z}{s} \right)^2, \quad j \neq 0$$

(16)

where only the first two non-zero terms have been kept. The sub(super)script 0 represents the wire that has been moved, and $j$ is the distance in wire position of a neighbor. Moving a wire alters not only its charge, but also its neighbor’s. The only important parameters for the calculation of the coefficients are the wire radius, grid pitch, and plane distance. The coefficients are shown in Figure 2 for different $r/s$ values. For the BetaCage $r/s = 0.005$ and $L/s = 1$. Reading the numbers off of the figure, the values of the coefficients for the BetaCage are $a_2^{(0)} = -0.2$, $b_2^{(0)} = 0.4$, $a_1^{(1)} = -0.11$, and $b_2^{(1)} = 0.06$.

Figure 3 show the effects of displacing wire 0 on itself and its nearest neighbor. The largest affect on the gain is on the nearest neighbor by a lateral displacement. A 265 $\mu$m change in wire position causes a 10% change in gain of the nearest neighbor.

The wire centering of a 50 $\mu$m hole has a maximum displacement of 12.5 $\mu$m; for a 100 $\mu$m hole it is 37.5 $\mu$m. Assuming a feed-through positioning tolerance of 25 $\mu$m and that this is uncorrelated with the hole size, the gain variations are 1% and 1.7% for the 50 $\mu$m and the 100 $\mu$m hole respectively.

The displacements of a single trigger-grid anode wire will be slightly worse as while all the coefficients are the same as for the bulk grid, the gain is slightly higher. Figure 4 shows the effects of displacing wire 0 in the trigger anode grid. A 235 $\mu$m change in wire position causes a 10% change in gain of the nearest neighbor, which is slightly worse than the bulk-frame case. This implies that for the 100 $\mu$m hole the gain variations will be on the order of 1.9%.

5.1 Verification of Sauli

To verify the above calculation of a single-wire displacement, the process is repeated for the test case given in Sauli [2] starting on page 55. The parameters for Sauli’s case of $L/s = 4$ and $d/s = 0.01$ are $a_2^{(0)} = -0.18$, $b_2^{(0)} = 0.28$, $a_1^{(1)} = -a_1^{(-1)} = -0.16$, $b_2^{(1)} = -b_2^{(-1)} = 0.025$, and $a_1^{(2)} = -a_1^{(-2)} = -0.04$. These parameters are eye-balled from Figure 2 and may vary from the parameters chosen by Sauli. Figures 5 and 6 show a comparison with these values to the figures given in Sauli. The plots are reproduced fairly accurately, and
Figure 2: Coefficients for the calculation of a single-wire displacement. Plots taken from Ref. [3] Figures 4 and 5.

The gain change for displacement of wire 0 in both the $x$ (lateral) and $z$ (vertical) dimensions for both wire 0 and its nearest neighbor for the bulk grid.

any variations are simply due to slightly different choices in parameters. For a gain of $10^6$ and the parameters given above, it takes a 0.091 mm change in lateral position to produce a 10% gain variation (according to Sauli's gain-variation formula; Ref. [2], Eqn. 42). This is consistent with his claim of a 0.1 mm variation.

The major difference between Sauli’s case and the BetaCage appear to be the choice of $s$, and the gain; however, as Sauli’s gain formula is off by about a factor of two, our gain variation happens to roughly match.
Figure 4: The gain change for displacement of wire 0 in both the $x$ (lateral) and $z$ (vertical) dimensions for both wire 0 and its nearest neighbor for the trigger grid.

Figure 5: The charge variation for a lateral displacement of wire 0 as taken from Sauli [2], Fig. 60(a) (left) and as calculated using the coefficients eye-balled from Figure 2 (right). The figures are nearly identical.

As the charge variation scales linearly with fractional lateral displacement, the difference of $5/2$ in $s$ equates to the factor of $\sim 3$ allowable distance variation for the BetaCage over Sauli’s example.

A 6-plane Capacitance Matrix

The calculation of a joint bulk-trigger grid capacitance matrix is performed by adding two grids to Figure 1 above the grounded plane. The first grid, the trigger anode grid, is located at $\Delta z$, and a second grounded wire grid is located at $2\Delta z$. Additionally, $z_0$ is increased to 21.5 cm to keep a 20 cm drift region. The 6-plane
Figure 6: The charge variation for a vertical displacement of wire 0 as taken from Sauli [2], Fig. 60(b) (left) and as calculated using the coefficients eye-balled from Figure 2 (right). The figures are nearly identical.

The capacitance matrix is then

\[
C = \frac{\epsilon_0}{\text{mm}} \begin{pmatrix}
0.108 & -0.073 & -0.034 & -0.0003 & -0.00005 \\
-0.073 & 0.147 & -0.073 & -0.0006 & -0.0001 \\
-0.034 & -0.073 & 0.111 & -0.003 & -0.0005 \\
-0.0003 & -0.0006 & -0.003 & 0.114 & -0.066 \\
-0.00005 & -0.0001 & -0.0005 & -0.066 & 0.161
\end{pmatrix}
\]

where the two grids are nearly decoupled. The first three rows and columns are the same as the 3 x 3 case, the fourth row and column represents the inner-cathode grid, and the last row and column represents the trigger-frame anode grid. The gain on the trigger grid is $10^5$. Recalculating the gain on the bulk grid gives a few percent lower gain of $G = 10^{4.323}$ than the originally calculated $G = 10^{4.329}$.

References

